A combinatorial problem motivated by a data transmission application

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Abstract

A permutation array of length n is a set of permutations of n elements such that any two permutations coincide in at most one position. A general construction of permutation arrays is given. All permutation arrays of length 6 and maximal size 18 are determined. Bounds on cyclic arrays are given. An appendix describes the application to data transmission (due to A.J. Han Vinck).

Introduction

We consider n-tuples of distinct elements from some fixed set R of n elements, e.g. $R = \{1, 2, 3, \ldots, n\}$ (or equivalently, permutations of $123 \cdots n$). We will call such n-tuples just *tuples*. Let

$$S_n = \{123 \cdots (n-1)n, 123 \cdots n(n-1), \ldots, n(n-1) \cdots 321\}$$

denote the set of all the $n!$ tuples.

By a *permutation array* of length n (for short: an n-array) we will mean a subset of S_n with the property that any two tuples coincide in at most one position.

Han Vinck [7] used permutation arrays in an application to data transmission over power lines. We describe this application in an appendix. As usual, one wants an array which is as large as possible for given parameters. Let P_n be the maximal size of an n-array. Deza and Vanstone [2] proved that for all n we have the simple upper bound: $P_n \leq n(n-1)$. They also showed that $P_n = n(n-1)$ if n is a prime power.

In this note we will give some results for permutation arrays. We give a general construction and look at some special cases. Further, we consider $n = 6$ in detail. It is known [2] that $P_6 = 18$. We determine all arrays of size 18. These results are obtained by a combination of analysis and computer search. Finally, we give results on cyclic permutation arrays.

A general construction

Theorem 1 *Let R be a ring (commutative with unity) of size n. Let U be the set of (multiplicative) units in R. Let V be a subset of U such that $v - v' \in U$ for all distinct $v, v' \in V$.*

Let

$$C = \{(v \cdot x + y \mid x \in R) \mid v \in V, y \in R\}.$$

Then C is an array of length n and size $n \cdot |V|$.

Proof: First, $(v \cdot x + y \mid x \in R)$ is a permutation of R since $vx + y = vx' + y$ implies $v(x - x') = 0$ and so $x - x' = v^{-1} \cdot 0 = 0$. Next, if $v \cdot x + y = v' \cdot x + y'$ where $v \neq v'$ (and $v, v' \in V$), then

$$x = (y - y') \cdot (v - v')^{-1},$$

that is, x is uniquely determined. QED

An application of Theorem 1 is as follows: Let $n = \prod_{i=1}^{r} p_i^{e_i}$ be the prime factorization of n and assume that

$$p^e = p_1^{e_1} < p_i^{e_i} \text{ for } i > 1.$$

Let

$$R = GF(p_1^{e_1}) \times GF(p_2^{e_2}) \times \cdots \times GF(p_r^{e_r})$$

(direct product). For $1 \leq i \leq r$, let γ_{ij}, $j = 1, 2, \ldots, p^e - 1$ be distinct non-zero elements of $GF(p_i^{e_i})$. Let

$$V = \{(\gamma_{1j}, \gamma_{2j}, \ldots, \gamma_{rj}) \mid 1 \leq j < p^e\}.$$

Then the conditions of Theorem 1 are satisfied. Hence we get an array of size $n(p^e - 1)$. We state this as a theorem.

Theorem 2 *Let p^e be the least prime power in the factorization of n. Then*

$$P_n \geq n(p^e - 1).$$

Transformations and automorphisms

There are $n!^2$ transformations obtained by a combination of a permutation of the positions and a permutation (renaming) of the elements of the set R. The set S_n is invariant under all these transformations. Two arrays are *equivalent* if one is the image of the other under one of these transformations.

The automorphisms for an array C are the transformations which map C onto itself. The set of automorphisms is a group, the automorphism group $Aut(C)$. There are then $n!^2/|Aut(C)|$ arrays equivalent to C.

A classification of 6-arrays of size 18

It is known [2] that $P_6 = 18$. We will give a complete classification of the 6-arrays of size 18. In the process we give a new proof that $P_6 = 18$.

There are $\binom{6!}{18} \approx 3.4 \cdot 10^{34}$ subsets of S_6 of size 18. Hence a direct listing of these is infeasible. Some variation of backtracking seems more promising. What we have done is an initial analysis combined with a backtracking search.

We say that a 6-array C is of **type I** if some element appears in some position in at least four tuples of the array. We note that if an array of size 19 existed, then it would have been of type I.

We say that an array C is of **type II** if it has the property that for each position and each element, there are exactly three tuples in C having this element at this position. Clearly, an array of type II has size 18.

On arrays of type I

We will determine the 6-arrays of type I and size at least 18, up to equivalence. A array of type I is equivalent to some array containing 4 tuples with the element in the initial position. Let X be this set of 4 tuples. We call the initial element of the tuples in X for A. We write these 4 tuples as rows of a 4×6 matrix G. We note that we may permute the rows and permute the columns in G without loss of generality. We also note that the requirement for a 6-array implies that the elements in a column of G (except the first) are all distinct.

We consider two cases:

- Case 1: There exist a pair of tuples $Ax_2x_3x_4x_5x_6$ and $Ay_2y_3y_4y_5y_6$ in X, and two positions i and j such that $1 < i < j$, $x_i = y_j$ and $x_j = y_i$.
- Case 2: There are no such pair of tuples, that is, for all pairs of tuples $Ax_2x_3x_4x_5x_6$ and $Ay_2y_3y_4y_5y_6$ in X and pairs of positions i and j, such that $1 < i < j$, we have $x_i \neq y_j$ or $x_j \neq y_i$.

Case 1

Permuting the rows and columns of G if necessary we may assume that

$$G = G_1 = \begin{bmatrix} A & B & C & . & . & . \\ A & C & B & . & . & . \\ A & . & . & . & . & . \\ A & . & . & . & . & . \end{bmatrix}$$

Let us denote the one element which does not appear in the second column by F. Since F can appear at most once in column 3, at least one of rows 3 and 4 does not contain F in positions 3. We may assume that this is row 3 (interchanging the two rows if necessary). The matrix now has the form (since both D and E appear in the second column)

$$G_1 = \begin{bmatrix} A & B & C & . & . & . \\ A & C & B & . & . & . \\ A & D & E & . & . & . \\ A & E & . & . & . & . \end{bmatrix}$$

Permuting the last three columns if necessary, we may assume that

$$G_1 = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & . & . & . \\ A & D & E & . & . & . \\ A & E & . & . & . & . \end{bmatrix}$$

The element g_{24} in position $(2, 4)$ of G can not be A, B, or C since those elements already appear in the second row, nor D since D already appears in the fourth column. Hence, the possibilities for g_{24} are E or F. Similar arguments give the the following possibilities for the remaining positions of the second row:

j	g_{2j} can only be
4	E, F
5	D, F
6	D, E

We note that if $g_{24} = E$, then we must have $g_{26} = D$ and so $g_{25} = F$. Similarly, if $g_{24} = F$, then we must have $g_{25} = D$ and so $g_{26} = E$. Hence, we have two subcases:

$$G_{11} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & E & F & D \\ A & D & E & . & . & . \\ A & E & . & . & . & . \end{bmatrix} \quad G_{12} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & F & D & E \\ A & D & E & . & . & . \\ A & E & . & . & . & . \end{bmatrix}$$

Case 1, subcase 1

Considering the third row of G_{11} we see that we have the following possibilities:

j	g_{3j} can only be
4	B, C, F
5	B, C
6	B, C

This implies that we must have $g_{34} = F$, but we have two possibilities for the final two elements of row 3. This gives

$$G_{111} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & E & F & D \\ A & D & E & F & B & C \\ A & E & . & . & . & . \end{bmatrix} \quad G_{112} \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & E & F & D \\ A & D & E & F & C & B \\ A & E & . & . & . & . \end{bmatrix}$$

Similar reasoning shows that the fourth row is fixed in both cases:

$$G_{111} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & E & F & D \\ A & D & E & F & B & C \\ A & E & F & C & D & B \end{bmatrix} \quad G_{112} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & E & F & D \\ A & D & E & F & C & B \\ A & E & F & B & D & C \end{bmatrix}$$

Case 1, subcase 2

This subcase is completely analogous. Also in this case there are two possible choices for G:

$$G_{121} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & F & D & E \\ A & D & E & B & F & C \\ A & E & F & C & B & D \end{bmatrix} \quad G_{122} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & B & F & D & E \\ A & D & E & C & F & B \\ A & E & F & B & C & D \end{bmatrix}$$

Case 2

Case 2 is similar, but there is a new point to consider so we give some details. As in Case 1, we let F denote the element not in the second column. Let α be the element $\neq A$ which does not appear in the third column. Let $C \notin \{A, \alpha, F\}$ and such that the row with C in the second position does not contain F in the third position (such a C exists since there are at least 3 elements not in $\{A, \alpha, F\}$ which also appears in the second column, and for at most one of these F appears in the third position of the corresponding row). We move the rows with C in third, resp. second position to the top of G and the column with F in the top position to the far right, and get

$$G_2 = \begin{bmatrix} A & B & C & . & . & F \\ A & C & D & . & . & . \\ A & . & . & . & . & . \\ A & . & . & . & . & . \end{bmatrix}$$

By our choice of C and the fact that the condition of Case 2 is assumed, we have $B \neq D \neq F$. Moving the row with D in second position up to row 3 (if necessary) and permuting columns 4 and 5 if necessary, we get

$$G_2 = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & . & . & . \\ A & D & . & . & . & . \\ A & E & . & . & . & . \end{bmatrix}$$

We now consider the possibilities for the rest of row 2. Looking at the elements already in row 2 and in the various columns, we see that have the following possibilities for the remaining elements:

j	g_{3j} can only be
4	B, E, F
5	B, F
6	B, E

However, $g_{34} = B$ can be ruled out since this would imply that the second row is $ACDBFE$. But then $g_{15} = g_{26} = E$ and $g_{25} = g_{16} = F$ which is not possible by the condition of Case 2. Therefore, there are two possibilities for G:

$$G_{21} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & . & . & . & . \\ A & E & . & . & . & . \end{bmatrix} \qquad G_{22} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & F & B & E \\ A & D & . & . & . & . \\ A & E & . & . & . & . \end{bmatrix}$$

Case 2, subcase 1

Considering the third row of G_{21} we see that we have the following possibilities:

j	g_{3j} can only be
3	B, E, F
4	C, F
5	B, C
6	C, E

We see that each choice of g_{33} fixes the remaining three elements. Hence we have the following possible forms of G in this subcase:

$$G_{211} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & B & F & C & E \\ A & E & . & . & . & . \end{bmatrix} \quad G_{212} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & E & F & B & C \\ A & E & . & . & . & . \end{bmatrix} \quad G_{213} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & F & C & B & E \\ A & E & . & . & . & . \end{bmatrix}$$

For G_{211}, looking at rows and columns, we see that $g_{43} = F$. However, therefore there are no possibilities for g_{44}: B and C are excluded by the condition of Case 2, and D, E, and F already appears in column 4. We conclude that G_{211} can not be completed and so we do not have to consider this case further. For the other two matrices, the fourth row is uniquely determined (by the condition of Case 2 and the elements already present in row 4 and the various columns):

$$G_{212} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & E & F & B & C \\ A & E & F & B & C & D \end{bmatrix} \quad G_{213} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & E & F & B \\ A & D & F & C & B & E \\ A & E & B & F & D & C \end{bmatrix}$$

Case 2, subcase 2

This subcase is similar. For G_{22} there are 3 possible choices for row 3:

$$ADBEFC, \; ADECFB, \; ADFECB.$$

For the first of these, there are two possible choices for row 4, for the other two there is a unique choice for row 4. This gives the following four matrices to consider:

$$G_{221} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & F & B & E \\ A & D & B & E & F & C \\ A & E & F & B & C & D \end{bmatrix} \quad G_{222} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & F & B & E \\ A & D & B & E & F & C \\ A & E & F & C & D & B \end{bmatrix}$$

$$G_{223} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & F & B & E \\ A & D & E & C & F & B \\ A & E & F & B & C & D \end{bmatrix}, \quad G_{224} = \begin{bmatrix} A & B & C & D & E & F \\ A & C & D & F & B & E \\ A & D & F & E & C & B \\ A & E & B & C & F & D \end{bmatrix}$$

Search for the various cases

After our initial analysis we are therefore left with 10 possible matrices G. For each of these we have done a complete computer search to determine the maximal 6-array containing the four rows of G. Each search took of the order one hour (some less, some more) on a Sun Ultra 10. For a given G we first determined the set of tuples which are at distance 5 or 6 from each of the four rows of G; in the table below we denote the size of this set by $\nu(G)$. Then we used a backtracking algorithm to determine a maximal 6-array; in the table below we denote the size of the maximal array by $\mu(G)$. The results are summarized in the following table:

$\nu(G)$	174	171	171
$\mu(G)$	18	16	17
G	$G_{111},\ G_{112}$	G_{212}	$G_{213},\ G_{221}$
	$G_{121},\ G_{122}$	G_{224}	$G_{222},\ G_{223}$

Since $\mu(G) \leq 18$ in all cases, this computation proves in particular that $P_6 = 18$. We further see that only the first four matrices can be extended to arrays of size 18. The computation showed that there are exactly 6 inequivalent arrays of size 18 and type I. Six inequivalent arrays are given in the table below.

C_1^I	C_2^I	C_3^I	C_4^I	C_5^I	C_6^I
123456	123456	123456	123456	123456	123456
132564	132564	132564	132564	132564	132564
145623	145623	145623	145623	145623	145623
156342	156342	156342	156342	156342	156342
213645	213645	213645	213645	213645	213645
241536	251463	241536	251463	241536	264531
265314	264531	264351	316254	265314	316254
316254	316254	316254	354126	316254	354126
354126	354126	354126	365412	354126	365412
426513	365412	362415	426135	426513	426513
431652	426513	421365	431652	431652	453261
453261	431652	435216	463521	453261	461325
462135	536421	452631	524361	462135	536421
536421	541236	536421	541236	524631	541236
561243	562143	543162	562143	561243	562143
615432	625134	561243	614532	615432	625134
634215	634215	615432	635241	634215	634215
642351	642351	624513	642315	642351	642351

The arrays C_1^I and C_2^I have automorphism group of order 1, the other four of order 3. Hence the total number of arrays of type I is

$$2 \cdot \frac{720^2}{1} + 4 \cdot \frac{720^2}{3} = 1728000.$$

On arrays of type II

For arrays of type II we can do a similar initial analysis considering 3×6 matrices containing the 3 tuples with an initial element A. We skip the details and only give the result. The details can be found in [5].

The analysis shows that we only have to consider 4 matrices satisfying the conditions of case 1 and 8 matrices satisfying the conditions of case 2. Again we have done a complete search for arrays of type II containing one of these matrices. The search used the fact that for any partial array, any position and any element, this element appears at most 3 times in this position. Further, we have assumed without loss of generality that the first three tuples have 1 as the initial element, the next three have 2 as the initial element, etc. The complete search took less then one hour. The search showed that there are exactly 6 inequivalent arrays of type II. They are given in the table below.

C_1^{II}	C_2^{II}	C_3^{II}	C_4^{II}	C_5^{II}	C_6^{II}
123456	123456	123456	123456	123456	123456
132564	134562	134562	134562	134562	134562
145623	142635	142635	142635	142635	145623
231645	215364	216345	216345	216534	216534
256413	236415	235614	235614	235146	235146
264531	264153	254136	264153	264315	241365
314652	345126	345261	345126	345261	346152
352146	352461	356412	356412	356412	354216
365214	361245	362154	362541	362154	362541
436251	425613	426531	425361	425613	432615
451362	456132	451623	451623	451236	456321
463125	463521	463215	463215	463521	461253
516324	514236	512463	512463	513642	512463
524163	526341	531246	526134	521364	524631
541236	531624	564321	531246	546123	563124
612435	612543	613524	613524	614253	613245
625341	641352	625143	641352	631425	625314
643512	653214	641352	654231	652341	651432

The sizes of the automorphism groups for these six arrays are given in the following table.

	C_2^{II}, C_3^{II}	C_6^{II}	C_1^{II}, C_4^{II}, C_5^{II}
$\lvert Aut(C) \rvert$	24	36	60

We see that the total number of arrays of type II is

$$2 \cdot \frac{720^2}{24} + \frac{720^2}{36} + 3 \cdot \frac{720^2}{60} = 83520.$$

Cyclic arrays

In this last section we will give some results on cyclic arrays, that is, arrays with the property that for any tuple in the array, all cyclic shifts of this tuple also belongs to the array.

Let Π_n denote the maximal size of a cyclic n-array.

If we choose $R = Z_n$ in Theorem 1, we get a cyclic array. If p is the least prime factor of n, then $V = \{1, 2, \ldots, p-1\}$ satisfies the conditions of Theorem 1 and we get the following bound.

Theorem 3 *Let p be the least prime factor of n. Then*

$$\Pi_n \geq n(p-1).$$

If n is even, then $p = 2$ and Theorem 3 gives $\Pi_n \geq n$. We will show that we actually have equality in this case.

Theorem 4 *If n is even, then $\Pi_n = n$.*

Proof: Suppose that $\Pi_n > n$. Let C be a cyclic array of length n and size Π_n with elements in Z_n. Let $\mathbf{b} \in C$. Without loss of generality we assume that $b_i = i$ for all i (renaming the elements if need be).

Since there are exactly n cyclic shifts of $\mathbf{b}$, there is another tuple $\mathbf{a} \in C$ which is not a cyclic shift of $\mathbf{b}$. We show that the elements $(a_i - i)$ where $i = 0, 1, \ldots n-1$ are distinct (modulo n). Assume not, that is, there exists i and j where $0 \leq i < j \leq n-1$ such that $a_i - i \equiv a_j - j \pmod{n}$. Let $\mathbf{b}' = (b'_0, b'_1, \ldots, b'_{n-1})$ be the tuple obtained by a cyclic shift of $\mathbf{b}$ such that $b'_i = a_i$. Then

$$b'_j = b'_i + (j - i) \equiv a_i + j - i \equiv a_j \pmod{n},$$

and so $b'_j = a_j$, that is, $\mathbf{a}$ and $\mathbf{b}'$ coincide in both the positions i and j, a contradiction. Hence the elements $(a_i - i)$ are distinct modulo n. Therefore,

$$\frac{n(n-1)}{2} \equiv \sum_{i=0}^{n-1}(a_i - i) = \sum_{i=0}^{n-1} a_i - \sum_{i=0}^{n-1} i = \frac{n(n-1)}{2} - \frac{n(n-1)}{2} = 0.$$

Since $n(n-1)/2 \not\equiv 0 \pmod{n}$ when n is even, we have got a contradiction. Hence $\Pi_n = n$.

For $n = 2^m$ we have $\Pi_n = n$ and $P_n = n(n-1)$. This illustrates that Π_n / P_n may be arbitraryly small.

Appendix

This appendix is based on [3] and [7] and gives a short description of the application of permutation arrays to data transmission over power lines.

The main idea is to vary the voltage (by a small amount) and use the variation to transmit signals. There are three main forms of noise which may affect the transmission:

- Permanent narrow band noise, that is noise which affects some frequency over a long period (e.g. noise from computer and TV monitors).
- Impulse noise, that is noise of short duration which affects many frequencies.
- White Gaussian noise (background noise).

In many traditional data transmission media (e.g. telephone lines and satellite communication) white Gaussian noise is the dominating kind of error affecting the system, but in this application the other two kinds of error are more important . The modulation method suggested is n-FSK (frequency shift keying), see e.g. [6, p.130] with frequencies $f_i = f_0+(i-1)/T$ for $i = 1, 2, \dots, n$.

Let C be a permutation array of length n and size M, with elements from the set $R = \{f_1, f_2, \cdots, f_n\}$. A set of M symbols can be mapped (coded) into C. A tuple $(f_{i_1}, f_{i_2}, \dots, f_{i_n})$ is transmitted over n time slots, at time slot t, the signal corresponding to frequency f_{i_t} is transmitted. At the receiving end, for each frequency, the received signal is tested against a threshold. We illustrate with an example using the array C_1^I of length 6 described above. A tuple of this array is (431652), the corresponding tuple of frequencies is $(f_4 f_3 f_1 f_6 f_5 f_2)$. In the following square the frequencies used for transmission at each time slot are market with a star.

	1	2	3	4	5	6
f_1			*			
f_2						*
f_3		*				
f_4	*					
f_5					*	
f_6				*		

Time slots

Now, consider the receiving end. First we illustrate the situation when there has been no white Gaussian noise. A permanent narrow band noise above the threshold will be detected at all time slots. An impulse noise will similarly be detected for all frequencies at some time slot. For example, with narrow band noises at frequencies f_1, f_3 and f_4 and an impulse noise at time slot 5, the received signals above the threshold are given by the stars in the square below to the left. If we remove the stars that fill a complete row or a complete column (since most of these are the result of noise), we get the pattern below to the right.

	1	2	3	4	5	6
f_1	*	*	*	*	*	*
f_2					*	*
f_3	*	*	*	*	*	*
f_4	*	*	*	*	*	*
f_5					*	
f_6				*	*	

Time slots

	1	2	3	4	5	6
f_1						
f_2						*
f_3						
f_4						
f_5						
f_6				*		

Time slots

The sent tuple must therefore be of the form (x, x, x, f_6, x, f_2) where x denotes an *erasure*, that is, it can be filled in with any frequency. There can be at most one such tuple and there is one, namely the sent tuple $(f_4 f_3 f_1 f_6 f_5 f_2)$.

The effect of white Gaussian noise is to lower or increase the the signal at some frequency at some time slot, that is, delete a signal (remove a star) or insert a signal (add a star).

Suppose that the total number of errors of all kinds is at most $n-2$. Suppose that we are left with b non-erased positions after the initial analysis, that is, there are b stars in the diagram after removing complete rows and columns of stars. Suppose that a of these are correct and the remaining $b-a$ are due to white noise inserting a signal (inserting a star). The total number of errors due to narrow band noise, impulse noise, and white noise deleting signals is then $n-a$. The number of errors due to white noise inserting signals is therefore at most

$$(n-2)-(n-a)=a-2.$$

Hence $b-a \leq a-2$ and so

$$a \geq \frac{b}{2}+1.$$

Therefore, the sent tuple coincide with the received erased tuple in at least $b/2+1$ positions. There can not be another tuple in the array C_1^I which coincide with the received tuple (with erasures) in at least $b/2+1$ positions, because this tuple would coincide with the sent tuple in at least 2 positions. This shows that the coding scheme can correct any combination of $n-2$ or less errors.

We illustrate the general situation with one example. With the same sequence as before sent and the following noise: narrow band noise at frequency f_1, impulse noise at time slot 5, white noise deleting the signal at frequency f_2 for time slot 2, and white noise inserting a signal at frequency f_2 for time slot 3. The analysis of the received tuple gives $(f_4, x, f_3, f_6, x, f_2)$ with $b=4$ non-erased positions. The only array tuple in C_1^I which coincide with this tuple in at least $b/2+1=3$ non-erased positions is $(f_4 f_3 f_1 f_6 f_5 f_2)$.

References

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